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Proportionality in Liquid Democracy and Representative Democracy*

Leyla Ade[1], Matteo Michelini[2], and Pietro Vigiani[3]

[1] Universität Bayreuth, Bayreuth, Germany
[2] Eindhoven University of Technology, Eindhoven, Netherlands
[3] Scuola Normale Superiore, Pisa, Italy

Abstract. In this paper, we compare liquid democracy to representative democracy with respect to a *proportionality principle*, according to which agents with higher stakes should have more voting weight. We provide a formal model of voting systems that models agents' uncertainty towards a voting issue as influenced by stakes in the issue. We formalise the delegation process in representative democracy and liquid democracy and prove that only the latter satisfies the proportionality principle.

Keywords: Computational social choice · Formal political theory · Liquid democracy · Representative democracy · Proportionality.

1 Introduction

Liquid Democracy (LD) has received increasing academic interest in recent years as an alternative model of collective decision-making compared to standard representative systems [3, 17, 18]. Moreover, it was employed as a concrete decision-making system by several organizations and grassroots parties, like the Pirate Party in Germany [6] and the EU project WeGovNow [4]. The core of LD is that voters are allowed to choose whether to delegate their voting rights or to vote themselves, rather than defaulting their decisions to few (fixed) political representatives, as in Representative Democracy (RD). The large freedom of voters has put LD in the spotlight: does LD embody (better than the traditional forms) the principles that justify democracy?

In this paper, we argue that through this flexible model of representation, LD outperforms RD with respect to a formal notion of *proportionality*. The proportionality principle states that weight in any decision-making should be proportional to individual stakes, which measure how people's interests are affected by the options available in the decision ([5]).[4] Indeed, although equality

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[4] The concept of 'stakes' is purposefully left vague. As such, different factors can influence the way people's stakes are computed. For example, a person may have

(summed up by the principle 'one person, one vote') is often presented as one of the main criteria of a good and fair democracy [9], some authors propose to replace the principle of equality by a principle of proportionality [5, 16, 1]. Yet, distributing voting weight on the basis of interest can be a very demanding process if performed *a priori*, i.e. before the voting process begins. We argue that LD is able to solve this problem by taking the best of both worlds: it gives every voter equal voting weight while still enabling them to voluntarily create a more proportional voting scenario through delegation. Indeed, proportionality is generally a desirable feature of a democratic system, as it encourages participation and assures the protection of minorities, whenever they are more affected by a decision [16].

The paper is organised as follows. In Section 2 we describe our model of preference expression and interest-driven delegation; in Section 3 we prove our technical results and in Section 4 we evaluate these results in light of our theoretical assumptions and discuss some further lines of work.

Our contribution We provide a formal model to measure and assess proportionality in different voting systems, where agents are assigned numerical stakes and are able to delegate their vote. Through the delegation process, a notion of voting weight is defined to account for the agents' number of received delegations. Finally, we provide a notion of proportionality, measuring expected voting weight with respect to stakes. We show that under the assumption that stakes influence the expertise of the voters, LD satisfies the proportionality principle for any profile, while RD does not.

Related Work To our knowledge, our work is the only formal study of LD from the point of view of proportionality. A similar connection has been studied from an informal perspective by Valsangiacomo [19]. Examples of issues discussed about LD include game-theoretic aspects of delegation process [2, 11, 20] and its epistemic accuracy [7, 14]. The two most similar approaches are those of Green-Armytage [13] and Kahng, Mackenzie, and Procaccia [14]. The former adopts a similar, but metric, model in which voters know their values with error and LD is evaluated with respect to expressive loss in comparison to some form of RD. The latter studies LD's delegation mechanism with respect to direct voting instead of RD. Unlike both of them, however, we consider not just binary issues and we do not assume that there is a correct solution to the issue in question. Notably, alternative voting mechanisms, such as *storable votes* [8] and *quadratic voting* [15], have been proposed precisely to satisfy the proportionality principle. On the other hand, our aim is to study how independently motivated systems, like RD and LD, fare under such criterion.

high stakes because the area where she lives is affected by the decision. Or, she may have high stakes because the decision would increase her retirement age, and so on.

2 The model

In this section we describe our model and introduce our basic concepts of voting system, suitable profile, delegation process and a notion of expected proportionality.

Definition 1. *A* voting system *is a quintuple* $\mathcal{S} = \langle N, P, LP, A, \boldsymbol{S}, \rangle$ *with* $N = \{1, ..., n\}$ *set of agents;* $P, LP \subseteq N$ *sets of politicians and liquid politicians such that* $P \cup LP \subseteq N$, $P \cap LP = \emptyset$ *and* $P \cup LP \neq \emptyset$*:* $A = \{a_1, .., a_m\}$ *set of alternatives and* $\boldsymbol{S} = (S_1, ..., S_n)$ *stakes of each agents.*

Our starting point is a set of agents N who have to choose a policy among those available in A regarding a specific issue. P and LP represent, respectively, the set of politicians, who vote directly, and liquid-politicians, who can either vote themselves or delegate. RD and LD correspond to the special voting systems such that $LP = \emptyset$ and $LP = N$ respectively. Note that direct democracy is a special case of RD where $P = N$. Our definition of voting system also allows for the set $N \setminus (P \cup LP)$ to be non-empty. However, this set receives a substantive reading only if a particular voting system is specified. In particular, in RD-systems $LP = \emptyset$ and $N \setminus (P \cup LP)$ is just the set of the agents who cannot cast their vote, but only delegate; while in LD-systems $N \setminus (P \cup LP)$ is empty.

The vector $\mathbf{S}$ describes the stakes of the agents, so that for each $i \in N$ $S_i \in \{0, ..., |A| - 1\}$. Intuitively, agents' stakes represent the extent according to which agents are concerned with, or involved in, the voting issue. For the purpose of simplicity, we assume that there is a direct correspondence between agents' involvement and the effort they will spend in deciding their preferred alternative. As a result, agents with higher stakes will be more interested in determining what's best for them. This seems a reasonable simplification: agents that are affected more by a decision will delve deeper into the matter, they will be more inclined to spend time to evaluate the options, they will assess more carefully the advantages and disadvantages of each option, etc. For our purposes, all that matters is that higher stakes corresponds to greater effort, which in turn results in a more informed and precise assessment of each agent's subjectively best option. Indeed, we assume that each agent has an *unique best option*, and, consequently, the higher the stakes of the agent the more options the agent will be able to discard, as not beneficial to her.

We shall say that a voting system $\mathcal{S}$ is p-level when there are p many different values in $\mathbf{S}$, i.e. $p = |\{S_i\}_{S_i \in \mathbf{S}}|$. In addition, we call L_1 the set of agents with the lowest stakes, L_2 the set of agents with the second-lowest stakes and so on. In general,

$$L_0 = \emptyset$$
$$L_i = \{i \in N | S_i \leq S_j \text{ for all } j \in N \setminus (L_0 \cup \cdots \cup L_{i-1})\}.$$

Option set A_i describes i's preferences, i.e. it collects all the options that i considers equally good for her (they all seem to give the same amount of benefits to her). As mentioned, each agent has a unique best option, but she may not be able to single it out. The size of A_i depends on i's stakes, described by the vector of stakes $\mathbf{S}$ so that

$$|A_i| = |A| - S_i.$$

Hence, the higher i's stakes, the smaller the size of i's option set. The relationship between the number of options and agents' stakes formally captures the idea that agents who are more likely to put effort in the voting issue will end up with an opinion which is closer to their actual one, i.e. their option set contains fewer options besides the subjectively best one. Note that the way $|A_i|$ is determined is quite coarse-grained, as stakes are only allowed to range between 0 and $|A| - 1$. This means that our model is not able to differentiate between small differences in involvement in the voting issue. More precisely, it is only able to differentiate stakes insofar as they provide a means to discriminate between options, i.e. reduce the size of the option set. Example 1 gives an illustration of how citizens' stakes may be determined and how they relate with citizens' effort and expertise.

Example 1 (Building an airport). A group of citizens is asked to express their vote about the construction of a new airport. Imagine, in particular, that citizens are called to express a preference over the airport's number of strips, from one to four. Each citizen will have a preferred option, coming from their understanding of the issue. Suppose Alice lives at the opposite side of the city: her stakes in the decision are lower and, presumably, she will give less importance to the decision than Bob, who lives next to the airport. Therefore, it is likely that Alice will not put too much effort into understanding the matter and forming a specific opinion. She may get to think for example that four strips would cause a huge environmental damage, without discerning which of the other options benefits her the most. Bob, on the other hand, will spend great effort to make his decision and he will probably be able to distinguish his best option. For example, he may prefer one strip, so that from his house he will hear less noise.

We shall use Definition 1 to formalize this scenario. Let $\mathcal{S} = \langle N, P, LP, A, \mathbf{S}, \rangle$ describe the voting system for the vote on the number of strips such that $N \supseteq \{Alice, Bob\}$. If we assume all citizens in N to be asked to cast their vote directly, we obtain that $N = P$, hence a RD-system. On the other hand, some citizens could possibly be allowed to pass their vote to others, which would mean that $LP \neq \emptyset$ and $P \subsetneq N$. The set of alternatives contains all the possible numbers of strips, i.e. $A = \{1, 2, 3, 4\}$. Alice's stake is low $S_{Alice} = 1$, hence she could only eliminate one alternative from her personal option set $A_{Alice} = \{1, 2, 3\}$. Bob on the other hand has the highest stake possible $S_{Bob} = 3$ and could identify his unique best alternative, thus his option set is a singleton $A_{Bob} = \{1\}$.

Definition 2. *Let $\mathcal{S}$ be a voting system. A* M-suitable profile $\boldsymbol{A}_M = (A_{i_1}, ..., A_{i_m})$ *for $\mathcal{S}$ is a (partial) vector of option sets such that $m \leq n$ and for each $j \in M = \{i_1, \ldots, i_m\}$ we have that $|A_j| = |A| - S_j$. A* suitable profile*, denoted by $\boldsymbol{A}$ is a*

N-suitable profile. We call $\mathcal{A}_{\mathcal{S}}^{M}$ the set of all M-suitable profiles for $\mathcal{S}$ and $\mathcal{A}_{\mathcal{S}}$ the set of all suitable profiles for $\mathcal{S}$.

Definition 3. *Let $\mathcal{S}$ be a voting system. A M-suitable profile $\boldsymbol{A}_M$ for $\mathcal{S}$ is* compatible *with a suitable profile $\boldsymbol{A}$ iff for all $i \in M$ such that $A_i \in \boldsymbol{A}_M$ we have $A_i \in \boldsymbol{A}$. We shall denote as $\mathcal{A}_{\mathcal{S}}|\boldsymbol{A}_M$ the set of all suitable profiles for $\mathcal{S}$ that are compatible with $\boldsymbol{A}_M$.*

Notably, combining $\mathcal{S}$ with a suitable profile yields a voting system where agents have decided on their preferences. The notion of M-suitable profile and compatibility will turn out useful in the proofs, as we will have to work by combining partial profiles.

Having defined formally a setting which represents agents' preferences, we now propose a model of delegation that accounts for rational voters' behaviour in selecting their representatives. Note that a voting system represents a situation in which agents' option sets, thus their preferred alternative regarding the issue to be voted for, are not known yet. However, for the delegation process as we define it in Definition 4 all option sets need to be known.

Definition 4. *A delegation process is a function $d_{\mathcal{S},\boldsymbol{A}} : N \to P \cup LP$ such that:*

$$d_{\mathcal{S},\boldsymbol{A}}(i) = \begin{cases} i & \text{if } i \in P \\ j \;\; s.t. \quad \begin{array}{l} (1)\; j \in X = \operatorname*{arg\,min}_{k \in P \cup LP} \dfrac{|A_k \cap (A \setminus A_i)|}{|A_k|} \\ (2) |A_j| \leq |A_h| \text{ for all } h \in X \end{array} & \text{if } i \notin P. \end{cases} \tag{1}$$

We define the actual voting weight of i in $\mathcal{S}$ as $w_{\mathcal{S},\boldsymbol{A}}(i) = |\{j \in N | d_{\mathcal{S},\boldsymbol{A}}(j) = i\}|$. We shall denote with $D_i^{\boldsymbol{A}_M}$ i's possible delegation set, i.e. $D_i^{\boldsymbol{A}_M} := \{j \mid d_{\mathcal{S}}(j) = i$ for some $\boldsymbol{A} \in \mathcal{A}_{\mathcal{S}}$ suitable for $\mathcal{S}$ which is compatible with $\boldsymbol{A}_M\}$.

Ties are broken randomly. The idea is that delegation is always the result of a compromise between a set of formal criteria [3] and voters' available information (content criterion). We describe voter i's behaviour in two phases: in (1) i selects the agents more likely to vote for i's preferred option, while in (2) i selects the agents that have the smallest option sets, i.e. people with the greatest expertise on the matter, among those already selected[5]. The person a voter delegates to, so called *expert*[6], is the most convincing person she knows among those

[5] Note that, although LD generally allows for transitivity in delegations, it happens that transitivity is superfluous in this model, since by completeness of information all voters delegate their votes directly to an expert that does not delegate. This makes our model a *de facto* version of proxy voting [13].

[6] The problem of the choice of an expert has been widely addressed in the social epistemology literature (see e.g. Croce [10] and Goldman [12]). Here we mainly refer to the parameters discussed by Blum and Zuber [3], i.e. dialectical performance, absence of biases, track record of cognitive successes, etc.

(eligible for delegation) who share some of her basic ideas. Note that in RD, when $P = N$ we have that by Definition 4 everyone delegates to themselves, this is, everyone is a politician who votes directly. In this special case there occurs no proper delegation. We call $w_{\mathcal{S},\mathbf{A}}(i)$ *actual voting weight* to distinguish it from the notion of expected voting weight that will be defined in Definition 5. Besides the existence of a unique best option, note that we assume *sincerity*, i.e. voters communicate their true preferences, as well as *complete information*, hence for all agents i and j, i knows j's option set A_j. With these assumptions, agents are able to select their delegate in compliance with our content and formal criteria.

We shall extend Example 1, which described a group of citizens called to decide on the size of their airport, so as to include the modeling of the delegation process.

Example 2. In the scenario we described in Example 1 Alice is able to narrow her preferences down to four alternatives and Bob's option set is a singleton. Now, we consider three additional citizens, Charlie, Dylan and Eve, such that $N = \{Alice, Bob, Charlie, Dylan, Eve\}$. Both Charlie and Eve invest some time to reduce their option sets to a pair each, whereas Dylan has no stakes in the matter and is indifferent between all alternatives. Figure 2 illustrates this situation. Each row represents a level of stakes from the lowest ($S = 0$) to the highest ($S = 3$). In each row the dots represent the four alternatives respectively, a_1 being one strip, a_2 being two strips and so on. Each box illustrates an agents option set.

Suppose an LD-system is adopted and all the citizens are considered liquid politicians, i.e. $LP = \{Alice, Bob, Charlie, Dylan, Eve\}$. Take Alice as an example. By condition (1) of Equation 1, she excludes Dylan and Eve from her list of possible delegates, as they include option 4 in their option set. On the other hand, Eve's and Bob's option sets are included in Alice's and hence they both minimize $\frac{|A_k \cap (A \setminus A_{Alice})|}{|A_k|}$. The choice between Eve and Bob is then decided by condition (2), and since Bob's option set is strictly smaller than Eve's one, we get that $d_{\mathcal{S},\boldsymbol{A}}(Alice) = Bob$. It can be shown with analogue arguments that Dylan delegates to Bob as well, while Charlie, Eve and Bob vote directly.

If, on the other hand, RD is the voting system of choice, the delegation process could look very different. Assume, for example, that Dylan is the only politician, i.e. $P = \{Dylan\}$, $N \setminus (LP \cup P) = \{Alice, Bob, Charlie, Eve\}$ and $LP = \emptyset$. Consequently, everyone delegates to Dylan and for each agent i we get $d_{\mathcal{S},\boldsymbol{A}}(i) = Dylan$.

We now turn to the principle of proportionality, which states that the weight in any decision-making should be proportional to individual stakes. We use such a principle to evaluate a voting system based on the satisfaction of the proportionality requirement. To do so, we introduce the notion of probability of a suitable profile, and we identify the weight of an agent with the *expected voting weight*, i.e. the expected value of the actual voting weight. Therefore,

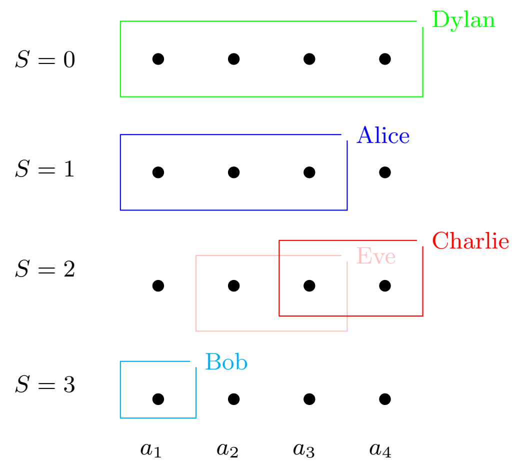

Fig. 1. Illustration of stakes and option sets of the airport Example 1 with more agents.

we define proportionality of a voting system in terms of a comparison between initial stakes and expected voting weight. Differently than actual voting weight, expected voting weight does not rely on the agents' actual option sets. Indeed, it is reasonable to judge whether or not a system is proportional before the preferences of the agents are known, as a voting system is often chosen in practice without such knowledge.

We assume that every suitable profile has the same probability to appear in $\mathcal{S}$, hence

$$P(\mathbf{A}|\mathcal{A}_{\mathcal{S}}) = p_{\mathcal{A}_{\mathcal{S}}}(\mathbf{A}) = \frac{1}{|\mathcal{A}_{\mathcal{S}}|}. \tag{2}$$

We define a random variable that takes on the voting weight of the different profiles that can be produced in $\mathcal{S}$.

Definition 5. *Let $\mathcal{S}$ be a voting system, $i \in N$ and $\mathcal{A}_{\mathcal{S}}$ be the set of all suitable profiles. The* voting weight $W^{\mathcal{A}_{\mathcal{S}}}(i) : \mathcal{A} \to \mathbb{N}$ *is a random variable over the sample space $\mathcal{A}_{\mathcal{S}}$ so defined:*

$$W^{\mathcal{A}_{\mathcal{S}}}(i)(\boldsymbol{A}) = w_{\mathcal{S},\boldsymbol{A}}(i).$$

Hence, i's expected voting weight is exactly the expected value of i's voting weight with respect to the voting system $\mathcal{S}$, i.e.

$$\mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(i)] = \sum_{\mathbf{A} \in \mathcal{A}_{\mathcal{S}}} w_{\mathcal{S},\mathbf{A}}(i) \cdot p_{\mathcal{A}_{\mathcal{S}}}(\mathbf{A}) = \sum_{\mathbf{A} \in \mathcal{A}_{\mathcal{S}}} w_{\mathcal{S},\mathbf{A}}(i) \cdot \frac{1}{|\mathcal{A}_{\mathcal{S}}|}. \tag{3}$$

Similarly, if we restrict the domain of the voting weight to range only on suitable profiles that are compatible with $\mathbf{A}_M$, we obtain that

$$\mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}|\mathbf{A}_M}(i)] = \sum_{\mathbf{A} \in \mathcal{A}_{\mathcal{S}}|\mathbf{A}_M} w_{\mathcal{S},\mathbf{A}}(i) \cdot \frac{1}{|\mathcal{A}_{\mathcal{S}}|\mathbf{A}_M|}. \tag{4}$$

Now that we have defined a notion of expected voting weight independently of the agents' preferences, we can formally define our proportionality principle for voting systems.

Definition 6. *Let $\mathcal{S}$ be a voting system and $\boldsymbol{A}$ a suitable profile for $\mathcal{S}$. Then $\mathcal{S}$ is* proportional *iff for all $i, j \in N$*

1. *if $S_i > S_j$ then $\mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(i)] > \mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(j)]$, and*
2. *if $S_i = S_j$ then $\mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(i)] = \mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(j)]$.*

Hence a voting system is *proportional* if and only if higher stakes lead to higher expected voting weight and equal stakes to equal expected voting weight accordingly. Two things are worth noting. First of all, our definition of proportionality implies that in a proportional voting system there may exist a suitable profile for which the actual voting weight (obtained after the delegation process) of an agent may be higher than the actual voting weight of an agent with higher stakes. Such a possibility should not sound strange, as our definition requires proportionality in terms of expected voting weight and not in terms of actual voting weight. Secondly, our definition draws a distinction between the case for equality and the case for 'greater than'. This trivially implies that if a system is proportional if $S_i \geq S_j$ then $\mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(i)] \geq \mathbb{E}[W^{\mathcal{A}_{\mathcal{S}}}(j)]$ for all $i, j \in N$.

3 Results

In this section we evaluate first RD and then LD with respect to proportionality. The following are two preliminary lemmas to the proof that the class of RD systems does not satisfy proportionality. Intuitively they state that unless the set of politicians consists of all agents except for the least interested ones, i.e. $P = N \setminus L_1$, a voting system can not be proportional. In what follows we omit reference to $\mathcal{S}$ when clear from context.

Lemma 1. *Let $\mathcal{S}$ be a p-level RD-system, i.e. $LP = \emptyset$. If $p > 1$ and $L_1 \not\subseteq N \setminus P$, then $\mathcal{S}$ is not proportional.*

Proof. By cases. Case 1: $L_1 \subseteq P$. If $N = P$, then $w_{\mathbf{A}}(i) = 1$ for every i and consequently, $E[W^{\mathcal{A}}(i)] = E[W^{\mathcal{A}}(j)]$ for every i, j for $S_i < S_j$. If $N \neq P$, then there exists j with $S_j \geq S_i$ such that $E[W^{\mathcal{A}}(i)] > E[W^{\mathcal{A}}(j)]$. Case 2: there exists $i \in L_1$ such that $i \notin P$. Thus, there also exists $j \in L_1 \cap P$ and then $w_{\mathbf{A}}(i) < w_{\mathbf{A}}(j)$ for any $\mathbf{A}$. In both the cases, $\mathcal{S}$ is not proportional. □

Lemma 2. *Let $\mathcal{S}$ be a p-level RD-system with $p > 1$. If $N \setminus P \nsubseteq L_1$, $\mathcal{S}$ is not proportional.*

Proof. By cases. Case 1: $L_1 \nsubseteq N \setminus P$. By the above Lemma $\mathcal{S}$ is not proportional. Case 2: $L_1 \subseteq N \setminus P$. So there exists $i, j \in N \setminus P$ such that $i \in L_1$ and $j \notin L_1$. Consequently, the expected voting weight for both is always 0, while they may have different stakes. ☐

Lemma 1 and Lemma 2 allow us to formulate our first theorem regarding RD-systems.

Theorem 1. *There is a triple $\langle N, A, \boldsymbol{S}\rangle$ such that for all possible set of politicians P, the RD-system $\mathcal{S} = \langle N, P, LP, A, \boldsymbol{S}\rangle$ is not proportional.*

Proof. Let $N = \{1, 2, 3\}, A = \{a_1, a_2, a_3, a_4, a_5\}$ and $\boldsymbol{S} = (1, 2, 3)$. We show that there is no choice of P that makes $\mathcal{S}$ proportional. By cases. Case 1: $P \neq \{2, 3\}$. Thus, by lemmas 1 and 2, $\mathcal{S}$ is not proportional. Case 2: $P = \{2, 3\}$. Now, consider an arbitrary 1-suitable profile $\mathbf{A}_{\{1\}}$ where A_1 is fixed and A_2 and A_3 are not. There exist 100 suitable profiles for $\mathcal{S}$ compatible with it. In 48 of them 3 gets the delegation from 1, i.e. $w_{\mathbf{A}}(3) = 2$ and $w_{\mathbf{A}}(2) = 1$. In the other 52 profiles, 2 gets 1's delegation, i.e. $w_{\mathbf{A}}(3) = 1$ and $w_{\mathbf{A}}(2) = 2$. This holds for all possible 1-suitable partial profiles $\mathbf{A}_{\{1\}}$. Therefore, $\mathbb{E}[W^{\mathcal{A}}(2)] > \mathbb{E}[W^{\mathcal{A}}(3)]$, as in more suitable profiles $d(1) = 2$ than $d(1) = 3$. Thus, there exists no set P such that the system is proportional. ☐

This is one of the two key findings of this work. Theorem 1 says that even if we were to select politicians after knowing the distribution of stakes and the option sets, it could be the case there is no choice of politicians for which the system is proportional, i.e. RD-systems do not guarantee proportionality.

Now, we shall move our attention to LD-systems. In order to prove our main theorem, we shall first prove Lemma 3.

Lemma 3. Let $\mathcal{S}$ be a LD-system, i.e. $LP = N$, and let $i, j \in N$. If $S_i > S_j$ then:

1. for all $\mathbf{A}_M$ with $M = N \setminus \{i, j\}$, $\mathbb{E}[W^{\mathcal{A}_\mathcal{S}|\mathbf{A}_M}(i)] \geq \mathbb{E}[W^{\mathcal{A}_\mathcal{S}|\mathbf{A}_M}(j)]$;
2. there is $\mathbf{A}_M$ with $M = N \setminus \{i, j\}$ such that $\mathbb{E}[W^{\mathcal{A}_\mathcal{S}|\mathbf{A}_M}(i)] > \mathbb{E}[W^{\mathcal{A}_\mathcal{S}|\mathbf{A}_M}(j)]$.

Proof. We consider an M-suitable profile with $M = N \setminus \{i, j\}$, and, consequently, from now on we drop reference to $\mathbf{A}_M$ in i's delegation set $D_i^{\mathbf{A}_M}$. First, note that by $S_i > S_j$ and the definition of $d_{\mathbf{A}}$ it follows $D_j \subset D_i$. Then i's expected voting weight in $\mathbf{A}_M$ corresponds to the sum of the expected voting weights each agent grants i with respect to $\mathcal{A}|\mathbf{A}_M$. Let $Ex_h(i)$ for $h \in M$ represent i's expected voting weight with respect to $\mathcal{A}|\mathbf{A}_M$ coming only from agent h delegating to i. This implies that

$$\mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(i)] = \sum_{h \in D_i} Ex_h(i).$$

Note that $Ex_h(i)$ is well defined since by design every voter delegates independently from other voters. Notably,

$$\mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(i)] - \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(j)] = \sum_{h \in D_j} (Ex_h(j) - Ex_h(i)) + \sum_{h \in D_i \setminus D_j} Ex_h(i). \quad (5)$$

The difference in expected general voting weight between i and j is given by the difference in weight that each agent h in $D_i \cap D_j$ gives to the two plus the weight each agent h in $D_i \setminus D_j$ gives to i. We now prove that for every $h \in D_j$, $Ex_h(i) - Ex_h(j) \geq 0$. We have that $Ex_h(i) = \frac{C_{|A_h|,|A_i|} \times C_{|A|,|A_j|}}{C_{|A|,|A_i|} \times C_{|A|,|A_j|}}$ and $Ex_h(j) = mod \times \frac{C_{|A_h|,|A_j|} \times (C_{|A|,|A_i|} - C_{|A_h|,|A_i|})}{C_{|A|,|A_i|} \times C_{|A|,|A_j|}}$ where $C_{x,y}$ is the number of combinations where x items must be assigned to y places. Since h is giving weight to j in some $\mathbf{A}$ this voting weight may not be a integer, but a fraction due to a possible tie. On the contrary, the weight being given to i is always a integer, since if there was a tie with i then h would have not been in D_j. Indeed, note that h is fixed, and if there was an l fixed such that $A_l \subseteq A_h$, then $D_j = \emptyset$. We can then assume $mod = 1$ without loss of generality. We can compute $Ex_h(i) - Ex_h(j)$ with $|A| \geq |A_h| \geq |A_j| > |A_i|$. Let $z = |A|, x = |A_h|, y = |A_j|$ and $|A_i| = w$. Consequently,

$$Ex_h(i) - Ex_h(j) = \frac{\binom{x}{w}\binom{z}{y} + \binom{x}{w}\binom{x}{y} - \binom{z}{w}\binom{x}{y}}{\binom{z}{w}\binom{z}{y}}.$$

In particular, $\binom{x}{w}\binom{z}{y} + \binom{x}{w}\binom{x}{y} - \binom{z}{w}\binom{x}{y}$ is equal to

$$\begin{aligned}
&= \frac{1}{w!y!} \cdot \left(\frac{x!}{(x-w)!} \cdot \frac{z!}{(z-y)!} + \frac{x!}{(x-w)!} \cdot \frac{x!}{(x-y)!} - \frac{z!}{(z-w)!} \cdot \frac{x!}{(x-y)!} \right) \\
&= \frac{x!}{w!y!} \cdot \left(\frac{z!}{(x-w)!(z-y)!} + \frac{x!}{(x-w)!(x-y)!} - \frac{z!}{(z-w)!(x-y)!} \right) \\
&= \frac{x!z!}{w!y!(z-w)!(x-y)!} \left(\frac{(z-w)!(x-y)!}{x-w)!(z-y)!} + \frac{x!(z-w)!}{(x-w)!z!} - 1 \right).
\end{aligned}$$

Since $\frac{(z-w)!(x-y)!}{(x-w)!(z-y)!} = \frac{(z-w)\cdot \ldots \cdot (z-y+1)}{(x-w)\cdot \ldots \cdot (x-y+1)}$ we have that $\frac{(z-w)!(x-y)!}{(x-w)!(z-y)!} \geq 1$. Thus, $Ex_h(i) - Ex_h(j) \geq 0$, and $\mathbb{E}[\mathrm{W}_{\mathcal{A}|\mathbf{A}_M}(i)] \geq \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(j)]$ for all $\mathbf{A}_M$. This concludes the proof of item 1. Now for item 2. Since $S_i \neq 0$, there is always a way of building $\mathbf{A}_M$ such that for all $k \in M$, $A_k \not\subset A_i$. In such a partial profile, $D_i \neq \emptyset$; in fact, i always delegates to herself and so $i \in D_i$. Hence, $\sum_{h \in D_i \setminus D_j} Ex_h(i) > 0$. Thus, by Equation 5, for some $\mathbf{A}_M$

$$\mathbb{E}[\mathrm{W}_{\mathcal{A}|\mathbf{A}_M}(i)] > \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(j)].$$

$\square$

Lemma 3 states that if i' stakes are higher than j's two consequences follow. First of all, i's expected voting weight is always greater than j's with respect to an M-suitable profile that does not assign an option set neither to i nor to j. Secondly, there always exists at least one M-suitable profile such that i's expected voting weight is higher than j's. Now we turn to our central theorem, which draws on the results of Lemma 3 and generalizes them to all the suitable profiles.

Theorem 2. *If $\mathcal{S}$ is a LD-system, then $\mathcal{S}$ is proportional.*

Proof. Assume $\mathcal{S}$ is a LD-system. If $S_i = S_j$ we can define a bijective function f mapping any suitable profile $\mathbf{A}$ for $\mathcal{S}$ to a suitable profile $\mathbf{A}'$ such that $A'_i = A_j$ and $A'_j = A_i$. Consequently, $|\{\mathbf{A} \in \mathcal{A} | w_{\mathbf{A}}(i) - w_{\mathbf{A}}(j) = h\}| = |\{\mathbf{A} \in \mathcal{A} | w_{\mathbf{A}}(i) - w_{\mathbf{A}}(j) = -h\}|$ for any $h \in \mathbb{N}$. Hence, $\mathbb{E}[W(i)] = \mathbb{E}[W(j)]$.

Assume $S_i > S_j$. Let $i \in N$ and $M \subseteq N$. Thus, by laws of probability

$$\mathbb{E}[W(i)] = \sum_{\mathbf{A}_M \in \mathcal{A}^M} \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(i)] \times p(\mathbf{A}_M).$$

That is, instead of computing the weight for each combination in $\mathcal{A}_{\mathcal{S}}$, we first compute the expected weight for each group of combinations having the same partial profile in common and then we do the weighted sum of these values with the probabilities of each partial profile being the weights. Thus, $\mathbb{E}[W^{\mathcal{A}}(i)] - \mathbb{E}[W^{\mathcal{A}}(j)]$ is equal to

$$\begin{aligned} &= \sum_{\mathbf{A}_M \in \mathcal{A}^M} \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(i)] \times p(\mathbf{A}_M) - \sum_{\mathbf{A}_M \in \mathcal{A}^M} \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(j)] \times p(\mathbf{A}_M) \\ &= \sum_{\mathbf{A}_M \in \mathcal{A}^M} (\mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(i)] - \mathbb{E}[W^{\mathcal{A}|\mathbf{A}_M}(j)]) \times p(\mathbf{A}_M). \end{aligned}$$

Hence, since $p(\mathbf{A}_M) > 0$, by Lemma 3 we conclude that $\mathbb{E}[W^{\mathcal{A}}(i)] - \mathbb{E}[W^{\mathcal{A}}(j)] > 0$. □

The results above show that the proportionality principle is always satisfied by LD but not by RD, as some RD-systems cannot be proportional, regardless of the set of politicians. For example, we showed in the proof of Theorem 1 that a system with three agents having three different levels of stakes, none of them equal to zero, is never proportional. On the other hand, LD guarantees proportionality for any system regardless of the stakes' profile. This implies that no information about $\mathbf{S}$ is needed for LD-systems to be proportional, as agents themselves shape the delegation process. Instead, in RD-systems agents do not have the same freedom: while in LD every agent can choose between delegation and vote, in RD system every agent only has one option. The result of Theorem 1 is a consequence of this feature. RD-systems are too rigid to be able to grant proportionality, as agents are unable to adapt their delegations to the option profile generated. If an agent is a politician, she always casts her vote even if there is someone that shares her ideas, but is more expert. In contrast, in LD-systems agents shape the delegations based on the option sets they select, and this grants them more flexibility and responsiveness to different scenarios. LD-systems exploit the knowledge of the agents of each others' option sets to create the delegations. In this sense, LD not only allows for proportionality, but does so starting from a situation in which each agent contributes equally to the delegation process. This also enables LD to produce a completely equal distribution of voting weight, where each agent votes directly, if each agent has the same stakes' value.

4 Conclusion

This paper proposes a way to assess proportionality in different delegation systems. Our theoretical results show that Liquid Democracy fares better than Representative Democracy with respect to the principle of proportionality, formalised as a probabilistic notion. Liquid Democracy always produces a proportional outcome, as it assigns more expected voting weight to agents more interested in the issue. In particular, it does so regardless of the specific assignments of stakes values in the system.

The results have, however, some conceptual limitations. First, proportionality is always obtained in terms of our representation of stakes, which is relatively coarse-grained. In fact, as S_i can only take as many values as there are alternatives, we may not be able to capture slight differences in stakes. An alternative modeling choice would be to allow stakes to take arbitrarily fine-grained values, e.g. over natural numbers, to be able to accommodate more differences between agents' stakes. However, this would imply that two agents with different stakes' values may have two option sets of the same size. Consequently, their difference in stakes would not be visible in any way to the other agents, who just know the option sets sizes (complete information). Secondly, complete information about other agents' preferences throughout the delegation process is a strong assumption. Our results are heavily dependent on the agents' ability to possibly find their perfect delegates. Finally, it seems relatively controversial that individuals are able to measure their stakes, adjust their effort to them, and convert that effort into knowledge about their best options. Therefore, a natural next step would be to relax these assumptions and see how the results change accordingly.

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